

# An Inversion Formula for Orlicz Norms and Sequences of Random Variables

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## Abstract

Given an Orlicz function  $M$ , we show which random variables  $\xi_i$ ,  $i = 1, \dots, n$  generate the associated Orlicz norm, *i.e.*, which random variables yield  $\mathbb{E} \max_{1 \leq i \leq n} |x_i \xi_i| \sim \|(x_i)_{i=1}^n\|_M$ . As a corollary we obtain a representation for the distribution function in terms of  $M$  and  $M'$  which can be easily applied to many examples of interest.

**Keywords:** Orlicz Norms, Random Variables

## 1 Introduction

Over the years, various new tools and techniques were developed in the study of order statistics, where the  $k$ -th order statistic of a statistical sample of size  $n$  is equal to its  $k$ -th smallest value. For general literature on order statistics see [2] or [4]. Order statistics are well known tools in non-parametric statistics and find application in wireless networks, compressed sensing and appear naturally in Banach space theory, when one considers sequences of random variables. They are used to study certain parameters associated to convex bodies and also provide a tool to compute the eigenvalue distributions of random matrices. Just to mention a few, see [3], [6], [5], [12] or [18]. Over the last years, there was put tremendous effort in giving estimates for the order of these quantities. Very precise results were obtained in [11] (see also the references therein).

In [7] the authors studied the important maximal case, *i.e.*, the case of the 1-st order statistic. Given independent identically distributed (iid) random

variables  $X_1, \dots, X_n$ , the new approach started there, was to give estimates for the quantity

$$\mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \quad (1)$$

in terms of Orlicz norms, where the Orlicz norm depends on the distribution of the  $X_i$ 's (definitions are given below). These estimates turned out to be quite useful, *e.g.*, recently, in [1] estimates for support functions of random polytopes were obtained using this approach.

The case of discrete random variables was first studied in [14] and [15]. To be more precise, the authors considered averages over permutations and obtained that for any  $x \in \mathbb{R}^n$  and  $a_1 \geq \dots \geq a_n > 0$

$$\frac{1}{n!} \sum_{\pi} \max_{1 \leq i \leq n} |x_i a_{\pi(i)}| \sim \|x\|_M, \quad (2)$$

where  $M$  is defined by  $M^* \left( \sum_{i=1}^k a_i \right) = \frac{k}{n}$  for  $k = 1, \dots, n$  and extended linearly, where  $M^*$  is the dual function of  $M$ . They also obtained an inverse result, *i.e.*, they proved that given an Orlicz function  $M$ , the choice

$$a_i = n \left( M^{*-1} \left( \frac{i}{n} \right) - M^{*-1} \left( \frac{i-1}{n} \right) \right), i = 1, \dots, n, \quad (3)$$

yields the estimate in (2).

Here, we prove the inverse result in the setting of (1), *i.e.*, given an Orlicz function  $M$ , we show how to choose the distribution of the random variables  $X_i$ ,  $i = 1, \dots, n$  so that (1) is equivalent to the Orlicz norm of the scalar sequence  $(x_i)_{i=1}^n$ . The result we obtain (especially Corollary 3.2) is in most cases easier to apply than (3), since it is easier to calculate the derivative instead of the inverse of the dual function (see definitions below). Our main result is the following:

**Theorem 1.1.** *Let  $M$  be an Orlicz function with right-hand derivative  $M'$  such that  $M'(0) = 0$ . We define*

$$Q(\cdot) = \int y \delta_{\frac{1}{y}}(\cdot) dM'(y).$$

*Whenever*

$$\int y dM'(y) < \infty, \quad (4)$$

*we get for all  $X_1, \dots, X_n$  independent, distributed according to  $Q(\cdot)/Q([0, \infty))$  and all  $x = (x_i)_{i=1}^n \in \mathbb{R}^n$*

$$c_1 \|x\|_M \leq \mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_M,$$

*where  $c_1, c_2$  are positive absolute constants.*

It was shown in [7] that for iid  $p$ -stable random variables  $X_1, \dots, X_n$ ,  $1 < p < 2$ , and for all  $x = (x_i)_{i=1}^n \in \mathbb{R}^n$

$$c_1 \|x\|_p \leq E \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_p,$$

*i.e.*, a  $p$ -stable distribution generates the  $\ell_p$  norm. Here,  $p$ -stable means that the characteristic function of  $X_i$  is given by  $\psi(u) = \exp(-|u|^p)$ . Furthermore, in [7] and [11] the authors proved that for independent, identically standard gaussian random variables  $X_1, \dots, X_n$ , and for all  $x = (x_i)_{i=1}^n \in \mathbb{R}^n$

$$c_1 \|x\|_M \leq \mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_M,$$

where

$$M(s) = \sqrt{\frac{2}{\pi}} \int_0^s e^{-\frac{1}{2t^2}} dt.$$

However, this obviously does not give the  $\ell_2$ -norm. In fact, the obtained Orlicz norm is far from that, because of the gaussian tails. At first this seems astonishing, since we deal with the case of a 2-stable distribution. For  $p = 2$ , one could expect to get the  $\ell_2$ -norm, since a sequence of independent gaussian random variables spans a subspace of  $L_p$  isometric isomorphic to  $\ell_2$ . On the other hand, from a probabilistic point of view this seems somehow natural, because a 2-stable distribution is very different from  $p$ -stable distributions for  $1 < p < 2$ , *e.g.*, existence of moments.

Therefore, the question arose which distribution generates the  $\ell_2$ -norm. The second and third named author gave an answer in [17]. However, this answer is limited to the special situation and it seems that it cannot be generalized. In fact, our main result here immediately yields that  $\log -\gamma_{1,p}$ ,  $1 < p < \infty$  distributed random variables generate a truncated  $\ell_p$ -norm and especially for  $p = 2$ , we obtain the  $\ell_2$ -norm.

## 2 Preliminaries

We recall that a function  $M : [0, \infty) \rightarrow [0, \infty)$  is called an Orlicz function, if it is convex,  $M(0) = 0$  and  $M(t) > 0$  for all  $t > 0$ . If there exists a  $t_0 > 0$  such that  $M(t) = 0$  for  $t \leq t_0$ , we call it a degenerated Orlicz function. The dual function  $M^*$  of an Orlicz function  $M$  is given by the Legendre transform

$$M^*(x) = \sup_{t \in [0, \infty)} (xt - M(t)).$$

Again,  $M^*$  is an Orlicz function and  $M^{**} = M$ . Hence,  $M^*$  uniquely determines  $M$ . In case  $M(t) = t^p$ ,  $1 \leq p < \infty$ , we just have  $\|\cdot\|_M = \|\cdot\|_p$ . For all

$x = (x_i)_{i=1}^n \in \mathbb{R}^n$ , we define the Orlicz norm  $\|\cdot\|_M$  by

$$\|x\|_M = \inf \left\{ t > 0 : \sum_{i=1}^n M\left(\frac{|x_i|}{t}\right) \leq 1 \right\}$$

and define the Orlicz space  $\ell_M^n$  to be  $\mathbb{R}^n$  equipped with  $\|\cdot\|_M$ . For more details and a thorough introduction to Orlicz spaces we refer the reader to [13] or [19].

We will use the notation  $a \sim b$  to express that there exist two positive absolute constants  $c_1, c_2$  such that  $c_1 a \leq b \leq c_2 a$  and use  $a \sim_p b$  in case the constants depend on some parameter  $p > 0$ .

Since the Lebesgue-Stieltjes integral allows integration by parts, we obtain the following result:

**Lemma 2.1.** *Let  $M$  be an Orlicz function (possibly degenerated) with right-hand derivative  $M'$ . Then, for all  $x \geq 0$ ,*

$$M(x) = \int_{[0, \infty)} (x - y)^+ dM'(y).$$

*Proof.* We use integration by parts. For all  $x \geq 0$

$$\begin{aligned} & \int_{[0, \infty)} (x - y)^+ dM'(y) \\ &= xM'(0) + \int_{(0, x]} (x - y)^+ dM'(y) \\ &= xM'(0) + (x - x)^+ M'(x) - (x - 0)^+ M'(0) - \int_{(0, x]} M'(y) d(x - y)^+ \\ &= \int_{(0, x]} M'(y) dy = M(x). \end{aligned}$$

□

This easy observation, however, forms the starting point for our inversion formula.

**Remark 2.2.** *This integral representation is of Choquet-type, since the functions  $x \mapsto (x - y)^+$  are (up to normalization) the extreme points of the convex cone of Orlicz functions.*

In [11] the authors proved the following theorem:

**Theorem 2.3.** *Let  $X_1, \dots, X_n$  be iid random variables and  $\mathbb{E}|X_1| < \infty$ . For all  $s \in \mathbb{R}_{\geq 0}$  let*

$$M(s) = \int_0^s \int_{\frac{1}{t} \leq |X_1|} |X_1| dP dt.$$

*Then, for all  $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ ,*

$$c_1 \|x\|_M \leq E \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_M,$$

*where  $c_1, c_2$  are positive absolute constants.*

The theorem says that given a certain distribution, one can generate a corresponding Orlicz norm, *e.g.*,  $p$ -stable random variables generate the  $\ell_p$ -norm. We would like to point out that  $M$  can also be written in the following way

$$M(s) = \int_0^s \left( \frac{1}{t} \mathbb{P}(|X| \geq \frac{1}{t}) + \int_{\frac{1}{t}}^{\infty} \mathbb{P}(|X| \geq u) du \right) dt. \quad (5)$$

### 3 The Inversion Formula

Before we give the proof of our main theorem, we want to make some remarks to get a better understanding. First of all, note that condition (4) is no restriction, because to define an Orlicz norm  $\|\cdot\|_M$ , it is enough to define an Orlicz function  $M$  on the interval  $[0, T]$ , where  $M(T) = 1$ , and then extend it linearly.

Furthermore, whenever  $M'(0) = 0$  we see that  $Q(\cdot) = \int y \delta_{\frac{1}{y}}(\cdot) dM'(y)$  obviously defines a measure on  $\mathbb{R}$ , and since (4) is no restriction,  $Q(\cdot)/Q([0, \infty))$  is a probability measure. The distribution function of  $Q(\cdot)$  is given by

$$F(x) = Q((-\infty, x]) = \int y \delta_{\frac{1}{y}}((-\infty, x]) dM'(y) = \int_{1/x}^{\infty} y dM'(y),$$

since  $\delta_{\frac{1}{y}}((-\infty, x]) = 1$  whenever  $\frac{1}{y} \in (-\infty, x]$ , i.e., if  $y \geq \frac{1}{x}$ . In the same way we obtain the tail distribution function  $\bar{F} = 1 - F$ , i.e., for all  $x > 0$ ,

$$\bar{F}(x) = \int_0^{\frac{1}{x}} y dM'(y), \quad (6)$$

where we restrict ourselves to the non-negative real line, since we defined our Orlicz functions on  $[0, \infty)$ .

We would also like to point out that in the setting of Theorem 1.1, for all  $Q$ -integrable function  $h$

$$\int h(x)dQ(x) = \int \int h(x)dQ_y(x)dM'(y), \quad (7)$$

where  $Q_y(\cdot) = y\delta_{\frac{1}{y}}(\cdot)$ ,  $y > 0$ . We will now prove the main theorem.

*Proof of Theorem 1.1.* For some  $y > 0$ , let  $Q_y(\cdot) = y\delta_{\frac{1}{y}}(\cdot)$ . Then, we have for all  $s \geq 0$ ,

$$\int_0^s \int_{\frac{1}{t} \leq x} xdQ_y(x)dt = y \int_0^s \int_{\frac{1}{t} \leq x} x d\delta_{\frac{1}{y}}(x)dt. \quad (8)$$

Now, whenever  $\frac{1}{t} \leq \frac{1}{y}$ , we obtain

$$\int_{\frac{1}{t} \leq x} x d\delta_{\frac{1}{y}}(x) = \frac{1}{y},$$

and 0 otherwise. So we have

$$\int_0^s \int_{\frac{1}{t} \leq x} xdQ_y(x)dt = \int_0^s \mathbb{1}_{\{\frac{1}{t} \leq \frac{1}{y}\}} dt = (s - y)^+. \quad (9)$$

Therefore, using (7), we obtain

$$\int_0^s \int_{\frac{1}{t} \leq x} xdQ(x)dt = \int \int_0^s \int_{\frac{1}{t} \leq x} xdQ_y(x)dt dM'(y).$$

Using equation (8), we get

$$\int \int_0^s \int_{\frac{1}{t} \leq x} xdQ_y(x)dt dM'(y) = \int (s - y)^+ dM'(y).$$

Lemma 2.1 yields

$$\int_0^s \int_{\frac{1}{t} \leq x} xdQ(x)dt = M(s).$$

Now, note that  $\int ydQ(y) < \infty$ . Therefore, by Theorem 2.3, there exist positive absolute constants  $c_1, c_2$  such that for all  $x = (x_i)_{i=1}^n$

$$c_1 \|x\|_M \leq E \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_M.$$

□

**Remark 3.1.** Keeping Remark 2.2 in mind, equation (9) shows that the extreme points of the convex cone of Orlicz functions correspond to the extreme points of the convex cone of probability measures - the Dirac measures. This is the key idea to obtain the inversion formula.

As a corollary we obtain the following useful representation of the tail distribution function  $\bar{F} = 1 - F$ .

**Corollary 3.2.** *Let  $M$  be a twice continuously differentiable Orlicz function satisfying (4) and  $M'(0) = 0$ . Then, for any random variable  $X$  distributed according to  $Q(\cdot)/Q([0, \infty)) = \int y \delta_{\frac{1}{y}}(\cdot) dM'(y)/Q([0, \infty))$ , and for all  $x > 0$ ,*

$$\bar{F}(x) = \bar{F}_X(x) = \int_0^{1/x} y M''(y) dy = \frac{1}{x} M' \left( \frac{1}{x} \right) - M \left( \frac{1}{x} \right).$$

In particular, for all  $x > 0$ ,

$$\bar{F}(x) = \int_x^\infty \frac{1}{y^3} M'' \left( \frac{1}{y} \right) dy.$$

*Proof.* By Theorem 1.1, we can represent  $M$  in the following way

$$M(s) = \int_0^s \int_{\frac{1}{t} \leq |x|} |x| dQ(x) dt,$$

where  $Q(\cdot) = \int y \delta_{\frac{1}{y}}(\cdot) dM'(y)$ . Equation (6) gives

$$\bar{F}(x) = \int_0^{1/x} y dM'(y).$$

Since  $M$  is twice differentiable

$$\int_0^{1/x} y dM'(y) = \int_0^{1/x} y M''(y) dy.$$

Integration by parts yields

$$\int_0^{1/x} y M''(y) dy = \frac{1}{x} M' \left( \frac{1}{x} \right) - M \left( \frac{1}{x} \right).$$

In particular, by a change of variables,

$$\bar{F}(x) = \int_x^\infty \frac{1}{y^3} M'' \left( \frac{1}{y} \right) dy.$$

□

Thus, in the case of a twice continuously differentiable Orlicz function, we have a continuous density function. Especially, we obtain the distribution function

$$F(x) = 1 - \left\{ \frac{1}{x} M' \left( \frac{1}{x} \right) - M \left( \frac{1}{x} \right) \right\}.$$

Furthermore, Corollary 3.2 says that for any  $x > 0$

$$\mathbb{P}(X \geq x) = \frac{1}{x} M' \left( \frac{1}{x} \right) - M \left( \frac{1}{x} \right),$$

where  $X$  is distributed according to  $Q(\cdot)/Q([0, \infty))$ .

Now we consider two examples to show how easily our result can be applied to obtain the distribution of the random variables so that they generate the given Orlicz norm. We start with the case of  $\ell_p$ -norms.

**Example 3.3.** *Let  $1 < p < \infty$  and  $M : [0, \infty) \rightarrow [0, \infty)$ ,  $t \mapsto t^p$ . Then, for all  $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ , we get*

$$\mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \sim_p \|x\|_p,$$

in the case that  $X_i$ ,  $i = 1, \dots, n$  are iid random variables with

$$\mathbb{P}(X_i \geq x) = px^{-p}$$

for all  $x > 0$ .

*Proof.* By Corollary 3.2 we know that the tail distribution function of  $X_i$ ,  $i = 1, \dots, n$ , is given by

$$\bar{F}(x) = \int_0^{\frac{1}{x}} y M''(y) dy,$$

where  $M(y) = y^p$ . Thus, we get

$$\bar{F}(x) = \int_0^{\frac{1}{x}} yp(p-1)y^{p-2}dy = px^{-p}.$$

Therefore, the distribution function  $F$  has to be of the form  $F(x) = 1 - px^{-p}$ , which is up to an absolute constant the same as the distribution function of a  $\log \gamma(p, 1)$ -distributed random variable. Using Theorem 2.3 it is easy to show that those random variables generate the  $\ell_p$ -norm (see [17]).  $\square$

We would also like to emphasize that it is important to know the distribution which yields the  $\ell_2$ -norm, since this is a main ingredient to obtain embeddings into  $L_1$  (see [16] and [20]).

The following example is in connection with gaussian random variables.



**Example 3.4.** Let  $M : [0, \infty) \rightarrow [0, \infty)$  be given by

$$M(s) = \sqrt{\frac{2}{\pi}} \int_0^s e^{-\frac{1}{2t^2}} dt. \quad (10)$$

Then, for all  $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ , we get

$$\mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \sim \|x\|_M,$$

in the case that  $X_i$ ,  $i = 1, \dots, n$  are iid random variables with

$$\mathbb{P}(X_i \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy$$

for all  $x > 0$ .

*Proof.* Again, to obtain the distribution of the random variables  $X_i$ ,  $i = 1, \dots, n$  which generate the norm given by (10), we use Corollary 3.2 and get

$$\bar{F}(x) = \int_x^\infty \frac{1}{y^3} M''\left(\frac{1}{y}\right) dy.$$

So an easy computation immediately yields that for any  $x > 0$

$$\mathbb{P}(X_i \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy.$$

Theorem 2.3 with the representation of  $M$  given in (5) shows that these random variables generate the Orlicz norm  $\|\cdot\|_M$ .  $\square$

The last example shows that in order to obtain the Orlicz norm given by the Orlicz function defined in (10), we can take iid standard gaussian random variables.

## References

- [1] D. ALONSO-GUTIERRÉZ, J. PROCHNO, *Estimating Support Functions of Random Polytopes via Orlicz Norms*, preprint, 2012
- [2] N. BALAKRISHNAN, A.C. COHEN, *Order Statistics and Inference*, New York, NY: Academic Press (1991)
- [3] A. COHEN, W. DAHMEN, R. DEVORE, *Compressed sensing and best  $k$ -term approximation*, J. Amer. Math. Soc. 22 (2009), 211-231

- [4] H. A. DAVID, H. N. NAGARAJA, *Order statistics*, 3rd ed., Wiley Series in Probability and Statistics. Chichester: John Wiley & Sons (2003)
- [5] O. GUÉDON, *Gaussian Version of a Theorem of Milman and Schechtman*, Positivity 1, No.1 (1997), 1-5
- [6] Y. GORDON, O. GUÉDON, M. MEYER, A. PAJOR, On the Euclidean sections of some Banach spaces and operator spaces, Math. Scandinavica 91 (2002), 247-268
- [7] Y. GORDON, A. LITVAK, C. SCHÜTT, E. WERNER, *Orlicz Norms of Sequences of Random Variables*, Ann. Probab., 30 (2002), 1833–1853
- [8] Y. GORDON, A. LITVAK, C. SCHÜTT, E. WERNER, *Geometry of spaces between zonoids and polytopes*, Bull. Sci. Math., 126 (2002), 733–762
- [9] Y. GORDON, A. LITVAK, C. SCHÜTT, E. WERNER, *Minima of sequences of Gaussian random variables*, C.R. Acad. Sci. Paris, Ser 1, Math., 340 (2005), 445-448
- [10] Y. GORDON, A. LITVAK, C. SCHÜTT, E. WERNER, *On the minimum of several random variables*, Proc. AMS, 134 (2006), 3665-3675
- [11] Y. GORDON, A. E. LITVAK, C. SCHÜTT, E. WERNER, *Uniform estimates for order statistics and Orlicz functions*, Positivity, to appear
- [12] V. A. MARCHENKO, L. A. PASTUR, *Distribution of eigenvalues in certain sets of random matrices*, Mat. Sb. (N.S.), 72 (1967), 407-535 (Russian)
- [13] M. A. KRASNOSELSKI, Y. B. RUTICKII, *Convex Functions and Orlicz Spaces*, P. Noordhoff LTD., Groningen (1961)
- [14] S. KWAPIEŃ, C. SCHÜTT, *Some combinatorial and probabilistic inequalities and their application to Banach space theory*, Studia Math., 82, 91–106, 1985
- [15] S. KWAPIEŃ, C. SCHÜTT, *Some combinatorial and probabilistic inequalities and their application to Banach space theory II*, Studia Math., 95(2), 14–154, 1989
- [16] J. PROCHNO, *The Embedding of 2-concave Musielak-Orlicz Spaces into  $L_1$  via  $\ell_2$ -Matrix-Averages*, preprint, 2012

- [17] J. PROCHNO, S. RIEMER, *On the Maximum of Random Variables on Product Spaces*, Houston J. Math., to appear, 2012
- [18] M. RUDELSON, *Lower estimates for the singular values of random matrices*, C. R. Math. Acad. Sci. Paris 342 (2006), no. 4, 247-252
- [19] M. M. RAO, Z. D. REN, *Theory of Orlicz Spaces*, Marcel Dekker (1991)
- [20] C. SCHÜTT, *On the embedding of 2-concave Orlicz spaces into  $L_1$* , Stillwater, Oklahoma and Kiel, Studia Math., 113(1), 1995

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